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EMBEDDING SPACES WITH UNCONDITIONAL BASES

BY

WILLIAM J. DAVIS[†]

ABSTRACT

Every super-reflexive space with an unconditional basis is isomorphic to a complemented subspace of a super-reflexive space with a symmetric basis.

Lindenstrauss [4] proved that if X has an unconditional basis, then X is isomorphic to a complemented subspace of a space Y with symmetric basis. Szankowski [5] extended the result to show that if X is reflexive, Y may be chosen to be reflexive. The purpose of this note is to give a proof of these results which also yields the further extension: if X is uniformly convex (super-reflexive), then Y may be chosen to be uniformly convex (super-reflexive). The technique is based on the construction in [1].

Let Φ denote the finitely supported (scalar) sequences, and let $B_{3/2}$ (resp. B_2) denote the unit ball in Φ with the $l_{3/2}$ (resp. l_2) norm. Consider the set $mB_{3/2} + (1/m)B_2$ in Φ and let $|\cdot|_m$ denote its Minkowski functional. Let W_m be the completion of $\langle \Phi, |\cdot|_m \rangle$. If X has monotonely normalized unconditional basis (x_n), for a sequence m_k, with $m_k \geqq 2^{-k}$, let

$$Z = (\Sigma W_{m_k})_X = \{(z_k) \mid z_k \in W_{m_k}, \ \|(z_k)\| = \|\Sigma |z_k|_{m_k} x_k\|_X < \infty\}$$

and

$$Y = \operatorname{diag} Z = \{(z_k) \mid z_k = z \quad \text{for all } k\}.$$

Since $m_k \geqq 2^{-k}$, it is easy to see that $Y \supset \{(x, x, x, \cdots) \mid x \in l_{3/2}\}$. The sequence $(x, x, x, \cdots)$ will be denote by $\tilde{x}$.

Lemma 1 is a consequence of lemma 1 of [1].

LEMMA 1. *The sequence* $(\tilde{\delta}_k)$, *where* $\delta_k = (\delta_{kl})_{l=1}^{\infty}$, *is a monotonely symmetric basis for* Y.

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LEMMA 2. *If X is reflexive (uniformly convex), then Z (and therefore Y) is reflexive (uniformly convex).*

PROOF. The "reflexive" part of this lemma is immediate. Day proved that if (W_n) is a uniformly uniformly convex sequence of Banach spaces, then $(\Sigma W_n)_X$ is uniformly convex whenever X is uniformly convex with monotonely unconditional basis. It is easy to see that the W_n's are uniformly uniformly convex. A proof appears in [2].

The next lemma also appears in [2].

LEMMA 3. *If X is super-reflexive with unconditional basis (x_n), then X can be equivalently renormed to be uniformly convex so that (x_n) is monotonely unconditional.*

We now choose a sequence (m_l) and a sequence (k_l) so that, if

$$K_l = \sum_{j=1}^{l} k_j \quad \text{and} \quad u_l = \left(2 \sum_{i=K_{l-1}+1}^{K_l} \delta_i\right) \Big/ k_l^{7/12},$$

then $\tilde{u}_l = (u_l, u_l, \cdots)$ is a sequence in Y equivalent to the basis (x_l) for X. Once this is accomplished, the main result is proved, since $[\tilde{u}_l]$ is norm one complemented in Y by conditional expectation [3].

We first compute the norm in W_m of $u = \Sigma_{i=1}^{k} \delta_i$:

$$| u |_m = \inf \{\lambda \mid u = u' + u'', \quad m \| u'' \|_{l_2}, \ \frac{1}{m} \| u' \|_{l_{3/2}} \leqq \lambda \}.$$

Let $u = u' + u''$ be any decomposition of u, and notice first that $\|\Pi_k u'\|_{l_2} \leqq \|u'\|_{l_2}$, etc., so that we may as well assume $u' = \Sigma_1^k \beta_i \delta_i$ (here Π_k denotes the natural projection of Φ onto $[\delta_1, \cdots, \delta_k]$). Further, if P_k is the collection of all permutations of $\{1, \cdots, k\}$ and if, for $\rho \in P_k$, $S_\rho(u') = \Sigma \beta_i \delta_{\rho(i)}$, then by symmetry of (d_j), we also have

$$\left\| \frac{1}{k!} \sum_{\rho \in P_k} S_\rho(u') \right\| \leqq \| u' \|$$

for each of the norms $\|\cdot\|_{l_2}$ and $\|\cdot\|_{3/2}$. Thus, we may assume that $u' = \alpha u$, $u'' = \beta u$ with $\alpha + \beta = 1$. Since $\| u \|_{l_{3/2}} = k^{2/3}$ and $\| u \|_{l_2} = k^{1/2}$, we are now able to conclude that $| u |_m = mk^{2/3}/(m^2 + k^{1/6})$ which, for fixed k, is maximized at $m = k^{1/12}$. There we have $| u |_{k^{1/12}} = \frac{1}{2} k^{7/12}$ (explaining the normalization of u_l above). For any m, $| u_l |_m \leqq m k_l^{-1/12}$, so that, if $m_l^{12} = k_l$ for all l,

$$\sum_{j=1}^{l-1} | u_l |_{m_j} \leqq \frac{1}{m_l} \sum_{j=1}^{l-1} m_j.$$

Also, since $| u_l |_m \leqq k_l^{1/12}/m$, we see that

$$\sum_{j=1}^{\infty} |u_l|_{m_{l+j}} \leqq \sum_{j=1}^{\infty} (m_l/m_{l+j}).$$

Thus, if (m_l) is a sequence satisfying

$$\frac{1}{m_l}\sum_{j=1}^{l-1} m_j + m_l \sum_{j=1}^{\infty} m_{l+j}^{-1} < 2^{-l-1},$$

if $k_l = m_l^{12}$ and if $z_l = (\delta_{il}u_l)_{i=1}^{\infty}$ in Z, then $\|z_l - \bar{u}_l\|_Z < 2^{-l-1}$ so (z_l) and $(\bar{u}_l)$ are equivalent basis sequences. It is easy to see that (z_l) is equivalent to the basis (x_n). This completes the proof.

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THE OHIO STATE UNIVERSITY
COLUMBUS, OHIO 43210 U.S.A.